Examinee's No.______________

Department of Systems Innovation,
Graduate School of Engineering, The University of Tokyo
2014 Entrance Examination Test and Answer Sheets

Mathematical Problems Designed to Test Ability to Think Logically

Monday, August 26, 2013   13:00 - 15:30

Notice:

1. Do not open this booklet until the start of the examination.

2. If you find missing or badly printed pages, please notify the supervisor.

3. Write your examinee number at top of this page. Do not write your examinee number or name anywhere else in this booklet.

4. Write the answers including the outline of your solutions below each question.

5. Blank sheets for calculation are distributed separately. After the examination, they must be returned.

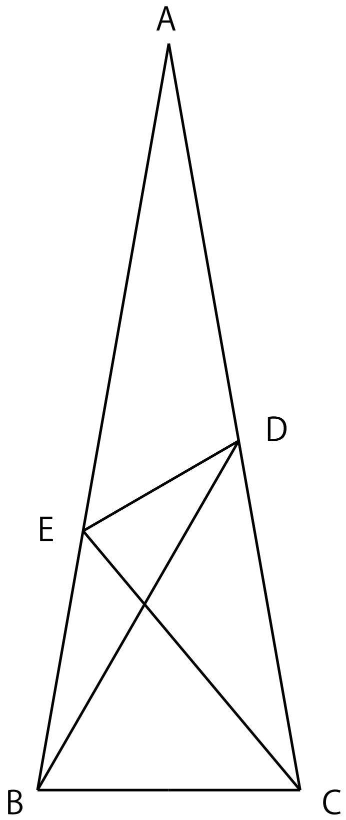
6. This test has 20 questions in total. Choose and answer 15 questions out of 20.
   <u>Check the question numbers of the questions you have selected in the table below. Please note that you are not allowed to answer more than 15 questions.</u>

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20

Q. 1

ABC is an isosceles triangle.

Assuming that $\angle BAC = 20°$, $\angle DBC = 60°$, and $\angle ECB = 50°$, obtain $\angle EDB$.



Q. 2

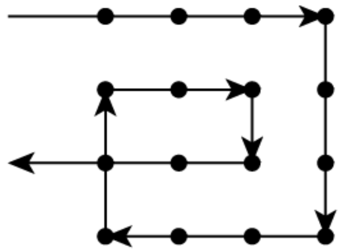
Given an $n \times n$ point lattice (where $n > 2$), connect all the points with $2n - 2$ straight lines without lifting your pencil from the paper.

Show the solutions for $n = 3$ and $n = 5$, indicating the direction of the lines.

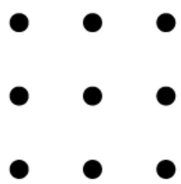
It should be noted that:

- you may pass through the same point more than once.
- lines may cross each other.
- you cannot redraw any portion of the same line.

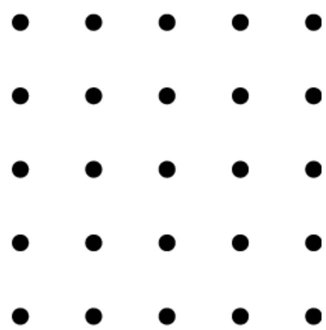
The figure below shows the wrong answer for $n = 4$, since it has seven lines instead of six as required.



Point lattices for your answer



$n = 3$



$n = 5$

Q. 3

Suppose $10^{0.30103} = 2$.

(1) Find how many digits are in 2^{100}.

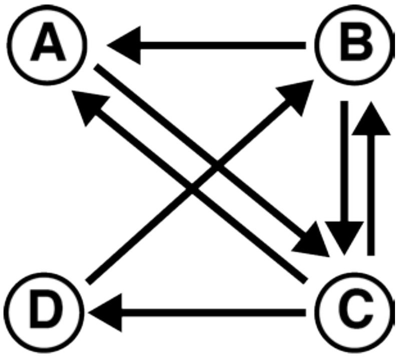
(2) At what decimal place does the first non-zero digit occur in 0.00025^{100}?

Q. 4

In the Cartesian coordinate system, the point P(2,1,0) was rotated by $\pi/3$ around the x-axis and then $\pi/4$ around the y-axis. There are two possible directions for rotating around each axis, and the direction of each rotation is not specified. Find the "new" coordinates of P after the rotations.

Q. 5

A company owns the following flight network (see the figure below) that connects the airports: A, B, C and D. Find the number of all permutations of flight routes to arrive at each airport (A, B and D) by taking no more than four flights, when he or she departs from airport C. Note that it is permissible to transit in and out of an airport more than once, including the airport designated as the final destination.



Q. 6

As shown in the following equation, a 9-digit natural number A divided by an 8-digit natural number B equals 89. Note that each □ represents a digit (i.e. 0, 1, 2, 3, 4, 5, 6, 7, 8, or 9), which can be used more than once.

Find A and B.

$$\frac{\mathrm{A}}{\mathrm{B}} = \frac{\square\square 3\ 4\ 5\ 6\ 7 \square\square}{\square\square\square 1\ 2 \square\square\square} = 89$$

Q. 7

Solve the following differential equation.

$$\frac{dy}{dx} + \frac{y}{x} = x^2 y^3$$

Q. 8

Find the value of c for the following matrix that has the rank of 2.

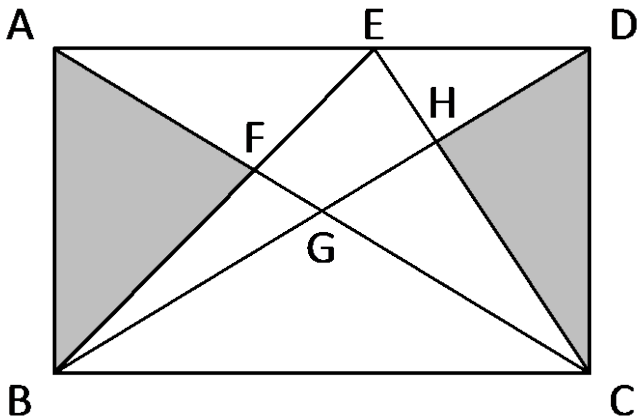
$$\begin{pmatrix} 1 & c & -1 & 2 \\ 2 & -1 & c & 5 \\ 1 & 10 & -6 & 1 \end{pmatrix}$$

Q. 9

There are n persons, each of which is in possession of a different piece of information. They want to share the information with each other by exchanging e-mails. Assume that each sender can include all pieces of information he or she knows at the time of sending the e-mail, which can be transmitted to only one recipient. What is the minimum number of e-mails they need to send in order to guarantee that everyone receives all pieces of information? Explain how you found the answer.

Q. 10

As shown in the figure, the rectangle ABCD has a height AB of 6 cm and a width BC of 9 cm. Calculate the area of a quadrangle EFGH, when the sum of the area of the two triangles ABF and CDH is 19 cm^2.



Q. 11

Find an equation for the tangent plane to the curved surface $z = 2\sqrt{6 - x^2 - y^2}$ at the point $(1, 1, 4)$.

Q. 12

How many arrangements for the first three digits of N^2 are possible, where $100000 \leq N^2 \leq 2500000$, and N is a natural number? It is worth noting that the arrangements for the first three digits of 1000000 and 1002001 are the same, and therefore they cannot be enumerated more than once.

Q. 13

100 students took a test where the perfect score is 100.

Show that there are at least two students who got the same score, when the summation of all students' score was 4900. Note that each score is a non-negative integer.

Q. 14

Every year, the emigration rate from country A to B is $\alpha\ (0 < \alpha < 1)$, whereas the emigration rate from country B to A is $\beta\ (0 < \beta < 1)$. Note that the fluctuation in population of both countries is only due to this migration, and that the total number of people living in countries A and B is always the same.

(1) Suppose that, in a certain given year, the population of countries A and B are X_0 and Y_0, respectively. Express the population of country A after 10 years as a function of X_0, Y_0, α, and β.

(2) Suppose that $\beta = 2\alpha$. Find the ratio between the populations of countries A and B, after an infinite number of years.

Q. 15

When a polynomial $P(x)$ is divided by $x^3+2x^2-13x+10$, the remainder is $2x+1$; in addition, when $P(x)$ is divided by $x^3+4x^2-15x-18$, the remainder is $2x^2-x-4$.
Find the remainder when $P(x)$ is divided by $x^3-19x+30$.

Q. 16

Find if the sum of the following infinite series either converges or diverges. The mathematical proof must be given.

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} + \cdots$$

Q. 17

A and B are points on the lines $y = x$ and $y = 2x$, respectively. If the length of the segment AB is 6, determine an equation for the trajectory of the point P which divides the segment AB internally in the ratio 2:1.

Q. 18

(1) What is the value of ‘Fa’ under the following conditions?

$\mathrm{Af} = -5,\ \mathrm{CC} = 6,\ \mathrm{pH} = -8,\ \mathrm{Hg} = 1,\ \mathrm{gk} = -18$

(2) What is the value of 12×6 under the following conditions?

$2 \times 3 = 6,\ 2 \times 4 = 11,\ 3 \times 3 = 12,\ 4 \times 5 = 26,\ 6 \times 5 = 42,\ 13 \times 14 = 215$

Q. 19

Somewhere in a town, there are two gas stations A and B, which are competing with each other. They purchase the same kind of gasoline, which price is 60 JPY/L. First, station A purchases x_A (L) of gasoline. Then, after knowing the quantity x_A, station B purchases x_B (L) of gasoline in order to maximize its profit. Note that the sale price P (JPY/L) of gasoline is determined according to the following formula, assuming that both stations sell all the gasoline they purchased.

$$P = \begin{cases} 200 - \dfrac{x_A + x_B}{1000} & (x_A + x_B \leq 140000) \\ 60 & (x_A + x_B > 140000) \end{cases}$$

Find the amount x_A that station A should purchase in order to maximize its profit.

Q. 20

You have nine matchsticks of the same length. Using all nine matchsticks, draw one figure that has three squares with the same area and two equilateral triangles with the same area.