Examinee's No._____

Department of Systems Innovation,
School of Engineering, The University of Tokyo
2013 Entrance Examination and Answer Sheets

Mathematical Problems Designed to Test
Ability to Think Logically

Monday, August 27, 2012   13:00 - 15:30

Notice:

1. Do not open this booklet until the start of the examination.

2. If you find missing or badly printed pages, please raise your hand to let the supervisor know.

3. Write your examinee's number on top of this sheet. Do not write your examinee's number or your name anywhere else in this booklet.

4. Write the answer including the outline of your solution below the question.

5. Sheets for calculation are distributed separately. After examination, they must be returned.

6. There are 20 questions in total. Choose and answer 15 questions out of 20.
    Check the question numbers of the questions you have selected in the bottom row below. Please note that you cannot answer more than 15 questions.

| 1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20

Q. 1

Suppose a coin-flipping game between two players, A and B. The probability that the coin lands heads up is p ($0<p<1$). In case a head appears, player A gets one point and player B loses one point. In case a tail appears, player B gets one point and player A loses one point. When either one of the players lose all their points, the game ends and the player having points becomes the winner. When the initial points of A and B are m and n respectively, calculate the probability that player A wins. Note that m and n are positive integers.

Q. 2

40 matchsticks of the same length are arranged as shown in the right figure. The figure includes 16 squares with a side of 1 matchstick; 9 squares with a side of 2 matchsticks; 4 squares with a side of 3 matchsticks; and 1 square with a side of 4 matchsticks. Now, remove 9 matchsticks so that there are no squares, and then draw the figure as a result. Note that you cannot move the remaining matchsticks to other places.
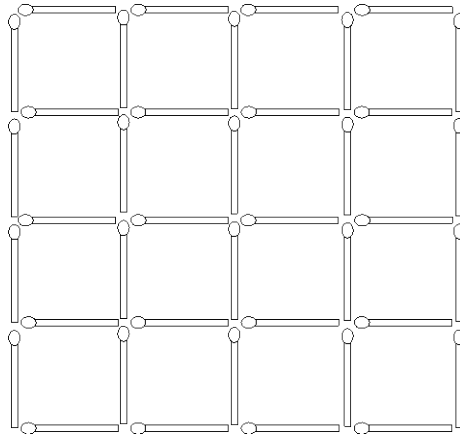
Q. 3

A traveler visited a village where an honest tribe and a liar tribe live. The honest tribe always tells the truth, and the liar tribe always tells lies. Answer the following questions.

(1) Five villagers are getting together, and each villager said the following:

> Villager A said, "Only one of us belongs to the liar tribe".
>
> Villager B said, "Two of us belong to the liar tribe".
>
> Villager C said, "Three of us belong to the liar tribe".
>
> Villager D said, "Four of us belong to the liar tribe".
>
> Villager E said, "All of us belong to the liar tribe".

Decide whether each villager is a member of the honest tribe or the liar tribe. Describe your reasoning.

(2) The traveler met two other villagers, and asked "Do you two belong to the same tribe?" From the answer of each villager, it is possible to judge their tribes. Describe your reasoning.

Q. 4

Solve the following equation. Note that x is a real number.

$$x^2 + 4x - 33 = -2\sqrt{x^2 + 4x + 15}$$

Q. 5

As shown in the figure, an object is created as an assembly of cubes by repeating the following procedure n times. The procedure starts from a cube with a side of length L.

> Procedure:
> For all cubes, create 27 smaller cubes by dividing each side equally into three segments. Then remove a smaller cube in the center and 6 smaller cubes in the middle of each face.

$n=0$ $n=1$ $n=2$
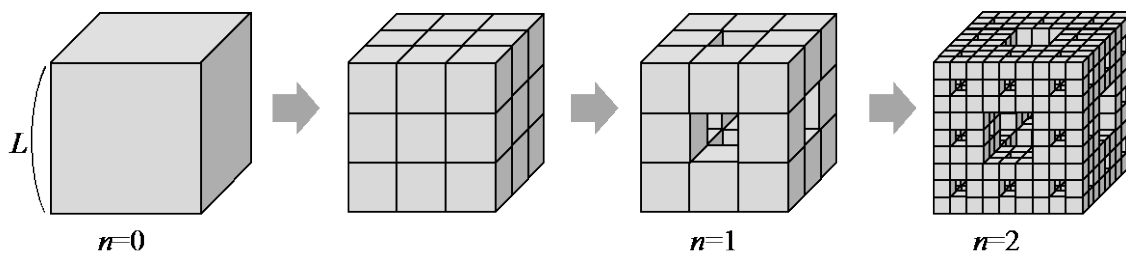
Answer the following questions.

(1) After repeating the procedure n times, calculate the volume of the created object.

(2) After repeating the procedure n times, calculate the surface area of the created object.

Q. 6

Supposing $\mathbf{P} = \begin{pmatrix} 1-a & a \\ b & 1-b \end{pmatrix}$ and n is a natural number, prove the following formula.

$$\mathbf{P}^n = \frac{1}{a+b}\begin{pmatrix} b & a \\ b & a \end{pmatrix} + \frac{(1-a-b)^n}{a+b}\begin{pmatrix} a & -a \\ -b & b \end{pmatrix}$$

Note that $a+b \neq 0$.

Q. 7

N is an integer and $2 \leq N \leq 700$. Answer how many Ns satisfy the condition that $N^2 - 1$ is a multiple of 280.

Q. 8

Obtain the volume of the intersection between two cylinders:
$x^2 + y^2 \leq 1$ and $y^2 + z^2 \leq 1$.

Q. 9

Obtain the general form of function $y(x)$ which satisfies the following equation, $y'' + 6y' + 9y = xe^{-2x}$.
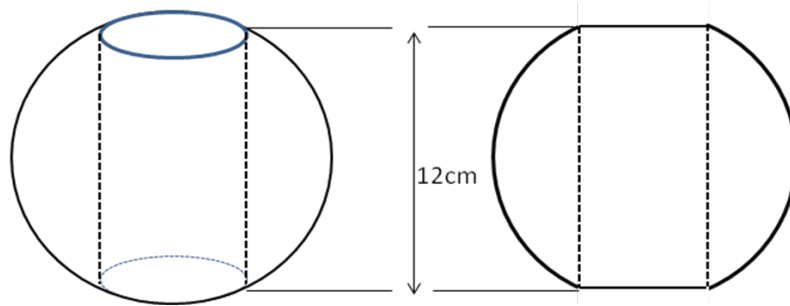
Q. 10

Obtain the integer numbers a, b, and c that satisfy the following equation,

$$6 \div 11 = \frac{1}{a} + \frac{1}{b} \times \frac{1}{c},$$

where $c > a > 0$ and $c > b > 0$.

Q. 11

Obtain the volume of the spherical ring that has a cylindrical hole with the height of 12cm. Note that the center of the sphere is on the central axis of the cylinder.

Q. 12

Answer the following questions about the equation: $r = a(1 + \cos\theta)$ $(a > 0)$.

(1) Draw the curve expressed by the equation in the coordinate system $(X, Y) = (r\cos\theta, r\sin\theta)$.

(2) Obtain the length of the curve.

Q. 13

There are two bags arranged in a row. One bag contains 2 white and 8 black balls and the other contains 6 white and 4 black balls. The first person picked a white ball out of the left bag and the second person picked a black ball out of the right bag. Note that each person picks only one ball and the balls which have been picked out are not returned to the bags.

Answer the following questions.

(1) When the third person picks a ball out of the left bag, obtain the probability that the ball is white.

(2) When the third person picks a ball out of the right bag, obtain the probability that the ball is white.

Q. 14

Answer the following questions. You may use the following formulas.

$$\sum_{m=1}^{n} m = \frac{n(n+1)}{2}, \quad \sum_{m=1}^{n} m^2 = \frac{n(n+1)(2n+1)}{6}, \quad \sum_{m=1}^{n} m^3 = \frac{n^2(n+1)^2}{4}$$

(1) Rewrite $m^5 - (m-1)^5$ as a fourth-order polynomial in m.

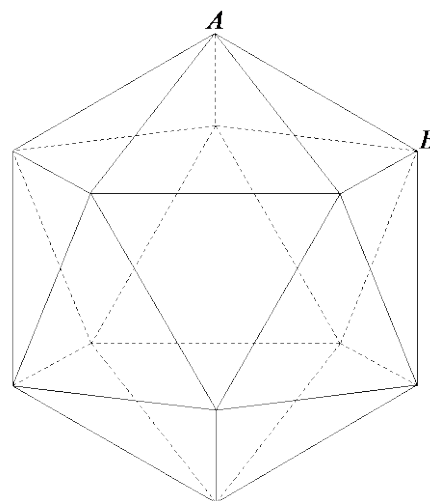(2) Find a formula for $\displaystyle\sum_{m=1}^{n} m^4$.

Q. 15

The following Cartesian coordinates define the vertices of a regular icosahedron (each face is an equilateral triangle) with edge-length 2: $(0, \pm 1, \pm p), (\pm 1, \pm p, 0), (\pm p, 0, \pm 1)$ (double sign in any order), where $p = \dfrac{\sqrt{5}+1}{2}$. The orthogonal projection of this icosahedron onto a plane containing one of its faces is depicted in the right figure. Obtain the edge-length AB of the outermost regular hexagon of the projection.

Q. 16

Let us make a formula equal to 10, using each of 4 numbers once in a given combination, arithmetic operations ($+-\times\div$), and parentheses. For example, $(7-2)\times6\div3$ is a formula for a combination (2, 3, 6, 7), and $(9\times9+9)\div9$ for (9, 9, 9, 9).

(1) Find a formula equal to 10 for a combination (1, 1, 9, 9).

(2) Find a formula equal to 10 for a combination (1, 1, 5, 8).

Q. 17

We want to measure the mass of any object less than or equal to 1kg in 1g increments with a balance. The object is put on one side and several weights on the other side. Find the minimum number of weights to be prepared and show one example of the combination of their weights. Note that you can use weights in 1g increments.

Q. 18

Obtain the following double integral within the bounded closed region, $D:\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}\le 1$ (x, y, a, and b are real numbers and $a>0$, $b>0$).

$$\iint_D \left(x^2+y^2\right)dx\,dy$$

Q. 19

Let x, u, and v be real numbers; a and τ be real constants. Solve the following simultaneous differential equations to obtain u and v.

$$\begin{cases} \dfrac{\partial^2 u}{\partial x^2} + 2a^2 v = 0 \\[2mm] \dfrac{\partial^2 v}{\partial x^2} - 2a^2 u = 0 \end{cases}$$

The boundary conditions are as follows.

$$\frac{\partial u}{\partial x} = 0, \ \frac{\partial v}{\partial x} = -\tau \ \left(\text{at } x = 0\right)$$

$$u = v = 0 \qquad \left(\text{at } x = \pm\infty\right)$$

Q. 20

E_1 to E_7 indicate single-digit even figures; O_1 to O_4 indicate single-digit odd figures. Find each figure to satisfy the following multiplication. Note that the same number may be used more than once.

$$
\begin{array}{rrr}
 & E_1 & O_1 \\
\times & O_2 & E_2 \\
\hline
 & O_3 & E_3 \\
E_4 & O_4 & \\
\hline
E_5 & E_6 & E_7 \\
\end{array}
$$